Rotating Strings with Two Unequal Spins in Lunin-Maldacena Background

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Abstract

We study a string motion in the Lunin-Maldacena background, that is, the β-deformed $AdS_5 \times \tilde{S}^5$ background dual to a β-deformation of $\mathcal{N} = 4$ super Yang-Mills theory. For real β we construct a rotating and wound string solution which has two unequal spins in $\tilde{S}^5$. The string energy is expressed in terms of the spins, the winding numbers and the deformation parameter. In the expansion of λ/J^2 with the total spin J and the string tension $\sqrt{\lambda}$ we present "one-loop" and "two-loop" energy corrections. The "one-loop" one agrees with the one-loop anomalous dimension of the corresponding gauge-theory scalar operators obtained in hep-th/0503192 from the β-deformed Bethe equation as well as the anisotropic Landau-Lifshitz equation.

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1 Introduction

The AdS/CFT correspondence [1] has been explored beyond the supergravity approximation [2, 3]. In the BMN limit [2] for the $\mathcal{N}=4$ super Yang-Mills (SYM) theory the perturbative scaling dimensions of gauge invariant near-BPS operators with large R-charge can be matched with the energies of certain string states, which has been interpreted as the semiclassical quantization of nearly point-like string with large angular momentum along the central circle of S^5 [3]. Moreover, the energies of various semiclassical string with several large angular momenta in $AdS_5 \times S^5$ have been shown in [4, 5, 6, 7, 8, 9, 10, 11] to match with the anomalous dimensions of the corresponding gauge invariant non-BPS operators which can be computed by using the Bethe ansatz [12] for diagonalization of the dilatation operator [13, 14, 15, 16, 17], that is represented by a Hamiltonian of an integrable spin chain. From the view point of integrability the gauge/string duality has been further confirmed by verifying the equivalence between the classical string Bethe equation for the classical $AdS_5 \times S^5$ string sigma model and the Bethe equation for the spin chain in the various sectors such as SU(2), SL(2), SO(6) and so [18, 19], while it has been demonstrated at the level of effective action where an interpolating spin chain sigma model action describing the continuum limit of the spin chain in the coherent basis was constructed [20, 21, 22].

In [23] Lunin and Maldacena have found a supergravity background dual to the Leigh-Strassler or β-deformation of $\mathcal{N}=4$ SYM theory [24] by applying a sequence of T-duality transformations and shifts of angular coordinates to the original $AdS_5 \times S^5$ background. They have taken the plane-wave limit of the deformed $AdS_5 \times \tilde{S}^5$ background with a deformed five-sphere $\tilde{S}^5$ and have shown that the string spectrum in the pp-wave coincides with the spectrum of BMN-type operators in the β-deformed $\mathcal{N}=4$ SYM [25].

The Lax representation for the bosonic string theory in the real β-deformed background has been constructed [26], which is similar to the undeformed case [27]. The gauge/string duality for the β-deformed case has been investigated by comparing the energies of semiclassical strings to the anomalous dimensions of the gauge theory operators in the two-scalar sector [28]. The energy of circular string with two equal angular momenta in $\tilde{S}^5$ has been computed from the semiclassical approach and has been also derived from the anisotropic Landau-Lifshitz equation for the interpolating string sigma model action which was obtained by taking the "fast-string" limit of the world sheet action in $AdS_5 \times \tilde{S}^5$. This interpolating action on the string theory side has been shown to coincide with the continuum limit of the coherent state action of an anisotropic XXZ spin chain on the β-deformed $\mathcal{N}=4$ SYM theory side, which was constructed using the one-loop dilatation operator in the deformed gauge theory [29, 30]. The classical string Bethe equation on the string theory side has been derived from the Lax representation of [26] to coincide with the thermodynamic limit of the Bethe equation for the anisotropic spin chain. Various relevant aspects of the gauge/string duality for the marginal deformed backgrounds have been investigated [31, 32, 33, 34, 35].

Multi-spin configurations of strings which move in both AdS_5 and $\tilde{S}^5$ parts of the β-deformed background have been constructed [32], where the winding numbers and the frequencies associated with the angular momenta take the same magnitudes respectively for each part of $AdS_5 \times \tilde{S}^5$. This property was also seen in the circular string solution with two equal spins [28] which is specified by the same frequencies and winding numbers. We

will construct a circular string solution with two unequal spins in $\tilde{S}^5$ which is characterized by different frequencies and winding numbers. The energy of the string solution will be computed and represented in terms of the unequal winding numbers, the unequal angular momenta and the deformation parameter. This energy spectrum on the string theory side will be compared with that of the solution in [28] for the anisotropic Landau-Lifshitz equation and the β-deformed Bethe equation for the spin configuration with the filling fraction away from one half.

2 Rotating string solution with two unequal spins

We consider a rotating closed string motion in the supergravity background dual to the real β-deformation of the $\mathcal{N}=4$ SYM theory. The β-deformed background for real $\beta=\gamma$ is given by [23]

$$\begin{aligned} ds^2_{str} &= R^2\left[ds^2_{AdS_5} + \sum_{i=1}^{3}(d\rho_i^2 + G\rho_i^2 d\phi_i^2) + \tilde{\gamma}^2 G\rho_1^2\rho_2^2\rho_3^2(\sum_{i=1}^{3} d\phi_i)^2\right], \\ B_2 &= R^2\tilde{\gamma}Gw_2, \qquad w_2 \equiv \rho_1^2\rho_2^2 d\phi_1 d\phi_2 + \rho_2^2\rho_3^2 d\phi_2 d\phi_3 + \rho_3^2\rho_1^2 d\phi_3 d\phi_1, \\ G^{-1} &= 1+\tilde{\gamma}^2 Q, \qquad Q \equiv \rho_1^2\rho_2^2 + \rho_2^2\rho_3^2 + \rho_3^2\rho_1^2, \qquad \sum_{i=1}^{3}\rho_i^2 = 1, \end{aligned} \tag{1}$$

where

$$(\rho_1,\rho_2,\rho_3) = (\sin\alpha\cos\theta, \sin\alpha\sin\theta, \cos\alpha), \qquad R^4 = Ng^2_{YM} = \lambda \tag{2}$$

and the regular deformation parameter $\tilde{\gamma}$ of the supergravity background is related with the real deformation parameter γ of the deformed $\mathcal{N}=4$ SYM as $\tilde{\gamma}=R^2\gamma$. We concentrate on a configuration that a closed string is staying at the center of AdS_5 and moving in the $\tilde{S}^3$ part of the deformed five-sphere defined by

$$\alpha = \frac{\pi}{2}, \qquad \text{i.e.} \qquad \rho_1 = \cos\theta,\ \rho_2 = \sin\theta,\ \rho_3 = 0. \tag{3}$$

The relevant bosonic string action takes the form

$$\begin{aligned} S \ = \ & -\frac{1}{2}R^2\int d\tau\int\frac{d\sigma}{2\pi}\Big[\gamma^{\alpha\beta}(-\partial_\alpha t\partial_\beta t + \partial_\alpha\theta\partial_\beta\theta + G\cos^2\theta\partial_\alpha\phi_1\partial_\beta\phi_1 + G\sin^2\theta\partial_\alpha\phi_2\partial_\beta\phi_2) \\ & -2\epsilon^{\alpha\beta}\tilde{\gamma}G\sin^2\theta\cos^2\theta\partial_\alpha\phi_1\partial_\beta\phi_2\Big], \end{aligned} \tag{4}$$

where $\epsilon^{01}=1, \gamma^{\alpha\beta}$ is expressed as $\gamma^{\alpha\beta}=\sqrt{-h}h^{\alpha\beta}$ in terms of a world-sheet metric $h^{\alpha\beta}$, and

$$G = \frac{1}{1+\frac{\tilde{\gamma}^2}{4}\sin^2 2\theta}. \tag{5}$$

We choose the conformal gauge $\gamma^{\alpha\beta}=\mathrm{diag}(-1,1)$ and make the following ansatz describing a closed string rotating and wound in the ϕ_1 and ϕ_2 directions

$$t=\kappa\tau, \qquad \phi_1=\omega_1\tau+m_1\sigma, \qquad \phi_2=\omega_2\tau+m_2\sigma, \qquad \theta=\theta_0=\text{const}, \tag{6}$$

where m_1, m_2 are the winding numbers. The string equation of motion for θ is satisfied when the constant θ_0 is specified by

$$\begin{aligned}\Big[\omega_1^2 - m_1^2 - (\omega_2^2 - m_2^2)\Big]\left(1 + \frac{\tilde{\gamma}^2}{4}\sin^2 2\theta_0\right) - 2\tilde{\gamma}(\omega_1 m_2 - \omega_2 m_1)\cos 2\theta_0 \\ + \tilde{\gamma}^2 \cos 2\theta_0 \Big[\cos^2\theta_0(\omega_1^2 - m_1^2) + \sin^2\theta_0(\omega_2^2 - m_2^2)\Big] = 0. \end{aligned} \tag{7}$$

It has a simple string solution with two equal angular momenta, which is described by $\theta_0 = \pi/4,\ \omega_1^2 - m_1^2 = \omega_2^2 - m_2^2$ in ref. [28]. Here we look for an extended solution with two unequal angular momenta which reduces to the simple solution in a special parameter limit. The equation (7) is rewritten in terms of $x = \cos 2\theta_0$ as

$$\frac{\tilde{\gamma}^2}{4}(\Omega_1 - \Omega_2)x^2 + \tilde{\gamma}\left(\frac{\Omega_1 + \Omega_2}{2}\tilde{\gamma} - 2\Omega_0\right)x + \left(1 + \frac{\tilde{\gamma}^2}{4}\right)(\Omega_1 - \Omega_2) = 0, \tag{8}$$

where $\Omega_i \equiv \omega_i^2 - m_i^2\ (i = 1, 2),\ \Omega_0 \equiv \omega_1 m_2 - \omega_2 m_1$. This equation determines x in terms of $\omega_i,\ m_i (i = 1, 2)$ and $\tilde{\gamma}$. The angular momenta $\mathcal{J}_1 = J_1/\sqrt{\lambda}$ and $\mathcal{J}_2 = J_2/\sqrt{\lambda}$ coming from the rotations in the ϕ_1 and ϕ_2 directions are obtained by

$$\mathcal{J}_1 = \frac{1}{1 + \frac{\tilde{\gamma}^2}{4}(1 - x^2)}\left[\frac{1+x}{2}\omega_1 + \frac{\tilde{\gamma}}{4}(1 - x^2)m_2\right], \tag{9}$$

$$\mathcal{J}_2 = \frac{1}{1 + \frac{\tilde{\gamma}^2}{4}(1 - x^2)}\left[\frac{1-x}{2}\omega_2 - \frac{\tilde{\gamma}}{4}(1 - x^2)m_1\right]. \tag{10}$$

From them the frequencies ω_1 and ω_2 are expressed as

$$\omega_1 = \frac{2}{1+x}\left[\left(1 + \frac{\tilde{\gamma}^2}{4}(1 - x^2)\right)\mathcal{J}_1 - \frac{\tilde{\gamma}}{4}(1 - x^2)m_2\right], \tag{11}$$

$$\omega_2 = \frac{2}{1-x}\left[\left(1 + \frac{\tilde{\gamma}^2}{4}(1 - x^2)\right)\mathcal{J}_2 + \frac{\tilde{\gamma}}{4}(1 - x^2)m_1\right]. \tag{12}$$

The conformal gauge constraints imply

$$\kappa^2 = \frac{1}{1 + \frac{\tilde{\gamma}^2}{4}(1 - x^2)}\left[\frac{1+x}{2}(\omega_1^2 + m_1^2) + \frac{1-x}{2}(\omega_2^2 + m_2^2)\right], \tag{13}$$

$$\frac{1+x}{2}\omega_1 m_1 + \frac{1-x}{2}\omega_2 m_2 = 0. \tag{14}$$

The substitution of ω_1 and ω_2 in (11) and (12) into the eq. (14) leads to a compact expression

$$\mathcal{J}_1 m_1 + \mathcal{J}_2 m_2 = 0. \tag{15}$$

It is noted that this expression for the γ-deformed background takes the same form as that for the undeformed background. When $\theta_0 = \pi/4$, that is, $x = 0$, the eqs. (9), (10) and (14) provide $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}/2,\ \omega_1 = \omega_2 = \mathcal{J} + \tilde{\gamma}(m + \tilde{\gamma}\mathcal{J}/2)/2$ with $m_1 = -m_2 \equiv m$ and the total

spin $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$, which is the special string solution with two equal angular momenta [28]. From (13) the energy of circular string solution is specified by

$$E^2 = \frac{\lambda}{1 + \frac{\tilde{\gamma}^2}{4}(1 - x^2)} \left[\frac{1+x}{2}(\omega_1^2 + m_1^2) + \frac{1-x}{2}(\omega_2^2 + m_2^2) \right]. \tag{16}$$

The quadratic equation (8) yields a solution expressed in terms of ω_1, ω_2, which is further inserted into (9), (10). If we can determine ω_1, ω_2 as functions of $\mathcal{J}_1, \mathcal{J}_2$ from the two inserted equations, we substitute these functions into (16) to obtain the energy expressed in terms of $\mathcal{J}_1, \mathcal{J}_2$. However, it is impossible to derive the functions so that we take the following alternative procedure. First combining (11) and (12) with (16) we have the energy expression

$$\begin{aligned} E^2 &= \lambda \Bigg[2\left(1 + \frac{\tilde{\gamma}^2}{4}(1 - x^2)\right)\left(\frac{\mathcal{J}_1^2}{1+x} + \frac{\mathcal{J}_2^2}{1-x}\right) + \tilde{\gamma}(-m_2\mathcal{J}_1 + m_1\mathcal{J}_2) \\ &+ \tilde{\gamma}(m_2\mathcal{J}_1 + m_1\mathcal{J}_2)x + \frac{1+x}{2}m_1^2 + \frac{1-x}{2}m_2^2 \Bigg]. \end{aligned} \tag{17}$$

When ω_1 and ω_2 in (11) and (12) are directly substituted into (8) we have an involved equation for x. If a solution x of the transcendental equation is obtained as a function of $\mathcal{J}_i, m_i (i = 1, 2), \tilde{\gamma}$ and inserted into (17), then the energy of string solution is expressed in terms of the angular momenta, the winding numbers and the deformation parameter.

3 Energy-spin relation

In order to solve the transcendental equation we consider the parameter region specified by $x \ll 1$ and take the expansion around $x = 0$. This is the case of almost equal spins, i.e. $\Delta J \ll J,\ \Delta J \equiv J_1 - J_2$. We use (11) and (12) to expand $\Omega_1 - \Omega_2$ in (8) in powers of x as

$$\Omega_1 - \Omega_2 = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots, \tag{18}$$

where

$$\begin{aligned} A_0 &= 4\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)\left[\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)(\mathcal{J}_1^2 - \mathcal{J}_2^2) - \tfrac{\tilde{\gamma}}{2}(m_2\mathcal{J}_1 + m_1\mathcal{J}_2) - \tfrac{1}{4}(m_1^2 - m_2^2)\right], \\ A_1 &= -8\left[\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)^2(\mathcal{J}_1^2 + \mathcal{J}_2^2) + \tfrac{\tilde{\gamma}}{2}\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)(-m_2\mathcal{J}_1 + m_1\mathcal{J}_2) + \tfrac{\tilde{\gamma}^2}{16}(m_1^2 + m_2^2)\right], \\ A_2 &= 12\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)\left(1 + \tfrac{\tilde{\gamma}^2}{12}\right)(\mathcal{J}_1^2 - \mathcal{J}_2^2) - 4\tilde{\gamma}\left(1 + \tfrac{\tilde{\gamma}^2}{8}\right)(m_2\mathcal{J}_1 + m_1\mathcal{J}_2) - \tfrac{\tilde{\gamma}^2}{4}(m_1^2 - m_2^2), \\ A_3 &= -16\left[\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)(\mathcal{J}_1^2 + \mathcal{J}_2^2) + \tfrac{\tilde{\gamma}}{4}(-m_2\mathcal{J}_1 + m_1\mathcal{J}_2)\right]. \end{aligned} \tag{19}$$

Similarly, $(\Omega_1 + \Omega_2)\tilde{\gamma}/2 - 2\Omega_0$ in (8) can be expanded as

$$\frac{\Omega_1 + \Omega_2}{2}\tilde{\gamma} - 2\Omega_0 = B_0 + B_1 x + B_2 x^2 + \cdots, \tag{20}$$

where

$$\begin{aligned} B_0 &= 2\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)^2\left[\tilde{\gamma}(\mathcal{J}_1^2 + \mathcal{J}_2^2) + 2(-m_2\mathcal{J}_1 + m_1\mathcal{J}_2) + \tfrac{\tilde{\gamma}}{4}\tfrac{m_1^2 + m_2^2}{1 + \frac{\tilde{\gamma}^2}{4}}\right], \\ B_1 &= -4\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)\left[\tilde{\gamma}\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)(\mathcal{J}_1^2 - \mathcal{J}_2^2) - \left(1 + \tfrac{\tilde{\gamma}^2}{2}\right)(m_2\mathcal{J}_1 + m_1\mathcal{J}_2) - \tfrac{\tilde{\gamma}}{4}(m_1^2 - m_2^2)\right], \\ B_2 &= 6\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)\left[\tilde{\gamma}\left(1 + \tfrac{\tilde{\gamma}^2}{12}\right)(\mathcal{J}_1^2 + \mathcal{J}_2^2) + \tfrac{2}{3}\left(1 + \tfrac{\tilde{\gamma}^2}{4}\right)(-m_2\mathcal{J}_1 + m_1\mathcal{J}_2) + \tfrac{\tilde{\gamma}^3}{48}\tfrac{m_1^2 + m_2^2}{1 + \frac{\tilde{\gamma}^2}{4}}\right]. \end{aligned} \tag{21}$$

It is noted that the coefficients A_k, B_k $(k = 0, 1, 2, \cdots)$ show the alternate expressions for $k =$ even and $k =$ odd. Hence by combining (18) and (20) with (8) and taking account of the behaviors that A_0 is of order ϵ, while A_1 and B_0 are of order ϵ^0, we make the leading order estimation of x as

$$x_1 = -\frac{\left(1+\frac{\tilde{\gamma}^2}{4}\right)A_0}{\tilde{\gamma}B_0 + \left(1+\frac{\tilde{\gamma}^2}{4}\right)A_1}, \tag{22}$$

which is rewritten by

$$x_1 = \frac{1+\frac{\tilde{\gamma}^2}{4}}{2(\mathcal{J}_1^2+\mathcal{J}_2^2)}\left[\mathcal{J}_1^2 - \mathcal{J}_2^2 - \frac{\tilde{\gamma}}{2}\frac{m_2\mathcal{J}_1 + m_1\mathcal{J}_2}{1+\frac{\tilde{\gamma}^2}{4}} - \frac{m_1^2 - m_2^2}{4\left(1+\frac{\tilde{\gamma}^2}{4}\right)}\right]. \tag{23}$$

Using the Virasoro constraint (15) and a parameter $\alpha = \mathcal{J}_1/\mathcal{J}$ with the total spin $\mathcal{J}$ we illustrate that x_1 is the term of order $\epsilon = 2\alpha - 1$

$$x_1 = \frac{2\alpha-1}{1+(2\alpha-1)^2}\left[1 + \frac{\tilde{\gamma}^2}{4} + \frac{\tilde{\gamma}}{2}\frac{m_1}{\mathcal{J}_2} + \frac{1}{4}\left(\frac{m_1}{\mathcal{J}_2}\right)^2\right]. \tag{24}$$

It is possible to estimate the non-leading term x_2 by substituting $x = x_1 + x_2$ into the eq. (8) accompanied with (18) and (20). By considering the behaviors that A_0, A_2 and B_1 are of order ϵ, while A_1, A_3, B_0 and B_2 are of order ϵ^0, we obtain

$$\begin{aligned} x_2 = -\frac{1}{\tilde{\gamma}B_0 + \left(1+\frac{\tilde{\gamma}^2}{4}\right)A_1}\Bigg[& x_1^2\left(\frac{\tilde{\gamma}^2}{4}A_0 + \tilde{\gamma}B_1 + \left(1+\frac{\tilde{\gamma}^2}{4}\right)A_2\right) \\ & + x_1^3\left(\frac{\tilde{\gamma}^2}{4}A_1 + \tilde{\gamma}B_2 + \left(1+\frac{\tilde{\gamma}^2}{4}\right)A_3\right)\Bigg], \end{aligned} \tag{25}$$

which is the term of order ϵ^3 as shown by

$$\begin{aligned} x_2 &= \frac{x_1^2}{\mathcal{J}_1^2+\mathcal{J}_2^2}\left[\frac{3}{2}\left(1-\frac{\tilde{\gamma}^2}{6}\right)(\mathcal{J}_1^2-\mathcal{J}_2^2) + \frac{\tilde{\gamma}^3}{8}\frac{m_2\mathcal{J}_1+m_1\mathcal{J}_2}{1+\frac{\tilde{\gamma}^2}{4}} + \frac{\tilde{\gamma}^2}{16}\frac{m_1^2-m_2^2}{1+\frac{\tilde{\gamma}^2}{4}}\right] \\ &- \frac{2x_1^3}{1+\frac{\tilde{\gamma}^2}{4}}. \end{aligned} \tag{26}$$

The substitution of $x = x_1 + x_2 + x_3$ where x_3 is the term of order ϵ^5 into the energy expression (17) yields the following expansion up to order ϵ^6

$$\begin{aligned} E^2 &= \lambda\Bigg[2(\mathcal{J}_1^2+\mathcal{J}_2^2) + \frac{1}{2}(m_2-\tilde{\gamma}\mathcal{J}_1)^2 + \frac{1}{2}(m_1+\tilde{\gamma}\mathcal{J}_2)^2 \\ &+ (x_1+x_2+x_3)\left(-2\left(1+\frac{\tilde{\gamma}^2}{4}\right)(\mathcal{J}_1^2-\mathcal{J}_2^2) + \tilde{\gamma}(m_2\mathcal{J}_1+m_1\mathcal{J}_2) + \frac{m_1^2-m_2^2}{2}\right) \\ &+ 2(\mathcal{J}_1^2+\mathcal{J}_2^2)(x_1^2+2x_1x_2+2x_1x_3+x_2^2+x_1^4+4x_1^3x_2+x_1^6) \\ &- 2(\mathcal{J}_1^2-\mathcal{J}_2^2)(x_1^3+3x_1^2x_2+x_1^5) + \cdots\Bigg]. \end{aligned} \tag{27}$$

Since the fourth term can be expressed as $(x_1 + x_2 + x_3)(-4(\mathcal{J}_1^2 + \mathcal{J}_2^2)x_1)$, we have

$$\begin{aligned} E^2 &= P + 2(J_1^2 + J_2^2)(x_1^4 + x_2^2 + 4x_1^3 x_2 + x_1^6) - 2(J_1^2 - J_2^2)(x_1^3 + 3x_1^2 x_2 + x_1^5) + \cdots, \quad (28) \\ P &\equiv 2(J_1^2 + J_2^2)(1 - x_1^2) + \frac{\lambda}{2}(m_2 - \gamma J_1)^2 + \frac{\lambda}{2}(m_1 + \gamma J_2)^2, \end{aligned}$$

where the $x_1 x_3$ term has been canceled out. The expression x_1 in (23) is rewritten by

$$x_1 = \frac{1}{1 + (2\alpha - 1)^2}\left[2\alpha - 1 + \frac{\tilde{\lambda}}{4}(m_2 - \gamma J_1)^2 - \frac{\tilde{\lambda}}{4}(m_1 + \gamma J_2)^2\right] \equiv \frac{\bar{x}_1}{1 + (2\alpha - 1)^2}, \tag{29}$$

where $\tilde{\lambda}$ is the effective coupling constant $\tilde{\lambda} = \lambda/J^2 = 1/\mathcal{J}^2$, so that P takes the form

$$\begin{aligned} P &= P_0 - \frac{\tilde{\lambda}^2 J^2}{16}\left[(m_2 - \gamma J_1)^2 - (m_1 + \gamma J_2)^2\right]^2 + J^2(2\alpha - 1)^2(1 + (2\alpha - 1)^2)x_1^2, \quad (30) \\ P_0 &\equiv J^2\left[1 - \frac{\tilde{\lambda}(2\alpha - 1)}{2}\Big((m_2 - \gamma J_1)^2 - (m_1 + \gamma J_2)^2\Big)\right] + \frac{\lambda}{2}(m_2 - \gamma J_1)^2 + \frac{\lambda}{2}(m_1 + \gamma J_2)^2. \end{aligned}$$

The leading part P_0 which is of order ϵ^0 is rearranged into

$$P_0 = J^2\left[1 + \frac{J_1 J_2}{J^2}\tilde{\lambda}(\gamma J + m_1 - m_2)^2 + \frac{\tilde{\lambda}}{J^2}(m_1 J_1 + m_2 J_2)^2\right], \tag{31}$$

which is expressed through the Virasoro constraint (15) as

$$P_0 = J^2 + J^2\alpha(1 - \alpha)\tilde{\lambda}(\gamma J + m_1 - m_2)^2. \tag{32}$$

Gathering together with the expression x_2 in (26) described by

$$\begin{aligned} x_2 &= \frac{x_1^2}{1 + (2\alpha - 1)^2}\bar{x}_2, \quad (33) \\ \bar{x}_2 &\equiv \left(1 - \frac{\tilde{\gamma}^2}{2}\right)(2\alpha - 1) + \tilde{\lambda}\gamma(m_2 J_1 + m_1 J_2) + \frac{\tilde{\lambda}}{2}(m_1^2 - m_2^2) \end{aligned}$$

we get the following energy expression

$$\begin{aligned} E^2 &= P_0 - \frac{\tilde{\lambda}^2 J^2}{16}\left[(m_2 - \gamma J_1)^2 - (m_1 + \gamma J_2)^2\right]^2 \quad (34) \\ &+ \frac{\tilde{\lambda}^2 J^2}{16}\frac{x_1^2}{1 + (2\alpha - 1)^2}\left[(m_2 - \gamma J_1)^2 - (m_1 + \gamma J_2)^2\right]^2 + 2J^2\bar{x}_1^2(2\alpha - 1)^4 + P_6 + \cdots, \end{aligned}$$

where

$$\begin{aligned} P_6 &\equiv J^2\bar{x}_1^4\Bigg[-2(2\alpha - 1)\bar{x}_1 + (2\alpha - 1)\tilde{\lambda}(\gamma J)^2\bar{x}_2 + \bar{x}_1^2 + \frac{\tilde{\gamma}^4}{4}(2\alpha - 1)^2 \\ &- \left(2\alpha - 1 + \tilde{\lambda}\gamma(m_2 J_1 + m_1 J_2) + \frac{\tilde{\lambda}}{2}(m_1^2 - m_2^2)\right)^2\Bigg]. \quad (35) \end{aligned}$$

The second term is of order ϵ^2 and the third term is of order ϵ^4 for the leading part, while the fourth term and P_6 are of order ϵ^6.

Now we use the resulting expression up to order ϵ^4 as well as up to order $\tilde{\lambda}^2$

$$E^2 = P_0 - \frac{\tilde{\lambda}^2 J^2}{16}\left[(m_2 - \gamma J_1)^2 - (m_1 + \gamma J_2)^2\right]^2 (1 - (2\alpha - 1)^2) + \cdots \tag{36}$$

to extract the energy of the solution as

$$\begin{aligned} E &= J + \frac{\lambda}{2J}\alpha(1-\alpha)(\gamma J + m_1 - m_2)^2 \\ &- \frac{\lambda^2}{8J^3}\alpha(1-\alpha)\left[\left((m_2 - \gamma J_1)^2 - (m_1 + \gamma J_2)^2\right)^2 + \alpha(1-\alpha)(\gamma J + m_1 - m_2)^4\right] + \cdots. \end{aligned} \tag{37}$$

The second term of order $\tilde{\lambda}$, that is, the "one-loop" energy correction agrees with the one-loop result of ref. [28], where the β-deformed Landau-Lifshitz action produced from "fast-string" expansion of the string sigma model action is shown to have a rational solution with two unequal momenta and the one-loop anomalous dimension is computed also by solving the Bethe equation for the corresponding anisotropic spin chain.

The undeformed limit $\gamma \to 0$ for (37) yields

$$E = J + \frac{\lambda}{2J}m(n-m) - \frac{\lambda^2}{8J^3}m(n-m)(n^2 - 3nm + 3m^2) + \cdots, \tag{38}$$

where the relations provided from (15) for the undeformed case

$$\frac{J_1}{J} = -\frac{m_2}{m_1 - m_2}, \qquad \frac{J_2}{J} = \frac{m_1}{m_1 - m_2} \tag{39}$$

have been used and the winding numbers m_1, m_2 have been replaced by $m_1 = n - m,\ m_2 = -m$. The reduced expression (38) including the "one-loop" and "two-loop" corrections was presented [18] by analyzing the classical Bethe equation for the classical $AdS_5 \times S^5$ string sigma model as well as the Bethe equation for the spin chain in the SU(2) sector. On the other hand in ref. [8] a general class of rotating string solutions in the undeformed background was derived and the "one-loop" correction in (38) was presented by solving the relations for the two spin sector

$$\mathcal{E}^2 = 2\sum_{i=1}^{2}\omega_i \mathcal{J}_i - \nu^2, \qquad \sum_{i=1}^{2}\frac{\mathcal{J}_i}{\omega_i} = 1, \qquad \sum_{i=1}^{2} m_i \mathcal{J}_i = 0, \tag{40}$$

where $\omega_i^2 - m_i^2 = \nu^2$ $(i = 1, 2)$ and the last equation indeed takes the same form as (15). Here we demonstrate that these relations also yield the "two-loop" correction. Indeed the following large $\mathcal{J}$ expansion

$$\mathcal{E}^2 = \mathcal{J}^2 + \sum_{i=1}^{2} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} + \frac{1}{4\mathcal{J}^2}\left[\left(\sum_{i=1}^{2} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}}\right)^2 - \sum_{i=1}^{2} m_i^4 \frac{\mathcal{J}_i}{\mathcal{J}}\right] + \cdots \tag{41}$$

leads to

$$E = \sqrt{\lambda}\mathcal{E} = J + \frac{\lambda}{2J}\sum_{i=1}^{2} m_i^2 \frac{J_i}{J} - \frac{\lambda^2}{8J^3}\sum_{i=1}^{2} m_i^4 \frac{J_i}{J} + \cdots . \tag{42}$$

The substitution of $m_1 = n - m,\ m_2 = -m$ and (39) into (42) reproduces (38).

Following the argument of [28], the structure of the energy of a multi-spin solution can be captured as double expansions in $\tilde{\lambda}$ and $\tilde{\gamma}$. The smoothness of the deformation gives

$$E = \sqrt{\lambda}\mathcal{E}(\tilde{\gamma}, \mathcal{J}) = \sqrt{\lambda}\left[\mathcal{E}_0(\mathcal{J}) + \tilde{\gamma} f_1(\mathcal{J}) + \tilde{\gamma}^2 f_2(\mathcal{J}) + \tilde{\gamma}^3 f_3(\mathcal{J}) + \tilde{\gamma}^4 f_4(\mathcal{J}) + \cdots\right], \tag{43}$$

while the energy for the undeformed case has the usual regular large $\mathcal{J}$ or small $\tilde{\lambda}$ expansion

$$E_0 = \sqrt{\lambda}\mathcal{E}_0(\mathcal{J}) = J f_0(\tilde{\lambda}) = J(1 + c_1\tilde{\lambda} + c_2\tilde{\lambda}^2 + \cdots). \tag{44}$$

Through $\tilde{\gamma} = \sqrt{\tilde{\lambda}}\gamma J$ the expansion (43) turns out to be

$$E = J\left[f_0(\tilde{\lambda}) + \tilde{\lambda}\gamma J f_1(\mathcal{J}) + \tilde{\lambda}(\gamma J)^2 \frac{f_2(\mathcal{J})}{\mathcal{J}} + \tilde{\lambda}^2(\gamma J)^3 f_3(\mathcal{J}) + \tilde{\lambda}^2(\gamma J)^4 \frac{f_4(\mathcal{J})}{\mathcal{J}} + \cdots\right]. \tag{45}$$

If $f_1(\mathcal{J}), f_2(\mathcal{J})/\mathcal{J}, f_3(\mathcal{J}), f_4(\mathcal{J})/\mathcal{J}$ have the regular expansions in $1/\mathcal{J}^2 = \tilde{\lambda}$ as

$$f_1 = \sum_{k=0}^{\infty} c_k^1 \tilde{\lambda}^k, \qquad \frac{f_2}{\mathcal{J}} = \sum_{k=0}^{\infty} c_k^2 \tilde{\lambda}^k, \qquad f_3 = \sum_{k=0}^{\infty} c_k^3 \tilde{\lambda}^k, \qquad \frac{f_4}{\mathcal{J}} = \sum_{k=0}^{\infty} c_k^4 \tilde{\lambda}^k, \tag{46}$$

then E also takes the following regular expansion

$$\begin{aligned} E &= J\Big[1 + \tilde{\lambda}\left(c_1 + c_0^1\gamma J + c_0^2(\gamma J)^2\right) \\ &+ \tilde{\lambda}^2\left(c_2 + c_1^1\gamma J + c_1^2(\gamma J)^2 + c_0^3(\gamma J)^3 + c_0^4(\gamma J)^4\right) + \cdots\Big]. \end{aligned} \tag{47}$$

Hence we can see that the "two-loop" part of order $\tilde{\lambda}^2$ contains the terms $(\gamma J)^n$ with $n \leq 4$. This behavior is indeed seen in our explicit "two-loop" result

$$\begin{aligned} E_2 &= -\frac{\lambda^2}{8J^3}\alpha(1-\alpha)\Big[\left((1-2\alpha)\gamma J + m_1 + m_2\right)^2 (\gamma J + m_1 - m_2)^2 \\ &+ \alpha(1-\alpha)(\gamma J + m_1 - m_2)^4\Big]. \end{aligned} \tag{48}$$

4 Conclusion

Analyzing the closed string motion in the γ-deformed $AdS_5 \times \tilde{S}^5$ background by the semiclassical approach, we have presented a transcendental equation which determines the energy of a solution describing a rotating and wound string with two unequal spins in $\tilde{S}^5$. By using an expansion procedure for the case of almost equal spins we have solved the transcendental equation to extract the string energy in terms of the angular momenta, the winding numbers and the deformation parameter.

The "one-loop" part of the string energy expanded in λ/J^2 has reproduced the one-loop anomalous dimension of long two-scalar operator which was found in ref. [28] as a solution

for the anisotropic Landau-Lifshitz and Bethe equations for the γ-deformed $\mathcal{N}=4$ SYM for the $SU(2)_\gamma$ sector. This agreement at the one-loop level confirms the gauge/string duality between the superstring theory in the γ-deformed $AdS_5\times\tilde{S}^5$ background and the γ-deformed $\mathcal{N}=4$ SYM theory. We have observed that the "two-loop" part of the string energy consists of the terms $(\gamma J)^n$ with $n\leq 4$, and recovers the corresponding expected "two-loop" string energy for the undeformed case when we take the undeformed limit $\gamma\to 0$. In order to confirm the gauge/string duality at the two-loop level it is desirable to construct some two-loop dilatation operator for the γ-deformed $\mathcal{N}=4$ SYM theory and compute the two-loop anomalous dimension of the relevant gauge invariant scalar operator.

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